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ON THE REPRESENTATION OF LARGE NUMBERS AND INFINITE PROCESSES

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IN a popular sense large numbers often cause amazement, or command respect and arouse curiosity. Sometimes, when occurring in problems of chance, they assure practical certainty, or reveal the law according to which an event may be repeated. From extremely large numbers we may pass to infinite numbers by defining these as numbers that are larger than any finite numbers. Although infinite numbers are no proper numbers at all and can not be explicitly represented or conceived, it is in many cases possible to state or establish the law of their formation, and to use them in the deduction of important results. In fact, they are of the greatest importance from a mathematical standpoint. The same is true of infinitely small numbers, the reciprocals of the infinitely large. Very small causes may have tremendous effects. But it is nevertheless true that, while theoretically the infinitely small is just as important as the infinitely large, it is the colossal, the enormity of a number, of a quantity, of a magnitude that appeals to the popular imagination. The distance of a thousand light years which light travels in a thousand years is, for example, of such a character. All astronomical distances, no matter how large, as long as they are defined by a finite number of light years, are finite and can be expressed by a finite number of miles. The infinitely small and large, on the other hand, are antinomies which can be approached through indefinitely increasing and decreasing finite numbers only; they are concepts which are imposed upon the human mind (*Zwangsvorstellung*) in the form of postulates. Throughout the history of human thought attempts have been made to explain the mystery of the infinite. In ancient times, sophistic and mystic methods, partly connected with the supernatural, were ordinarily followed to explain the secret. The well-known problem of Zeno is a typical example of the reasoning that was applied in case of an infinite process. The Eleat Zeno (born about 500 B.C.) contended that such things as multiplicity and motion were impossible. To prove that there can be no motion, Zeno devises the problem of Achilles. In a race between the fast runner Achilles and the slow moving tortoise, the latter can never be reached by Achilles, because the pursuer must always first reach the point where the pursued was. During this time-interval, the tortoise gains headway and

reaches a point ahead of the pursuer. Thus, during each successive time-interval the tortoise moves some distance ahead of Achilles, and as the number of intervals increases indefinitely, the race can never end. This, according to Zeno, proves the non-reality of motion. The sophistry of this reasoning is, of course, apparent, although there are some modern logicians who are inclined to side with Zeno.

Even much later as great a philosopher and mathematician as Leibniz (1646–1716) mixed up metaphysical speculations with the discussion of mathematical infinity. Leibniz discovered the diadic system, according to which every number may be represented by a definite succession of the two ciphers 0 and 1 only. He identified 1=unity with God and 0 with nothing. God created all things out of nothing, just as all integral numbers may be formed by the ciphers 0 and 1 in the diadic system. Leibniz was so enthusiastic over this seemingly remarkable discovery that he asked the Jesuite Grimaldi, at that time president of the mathematical tribunal of China, to communicate the discovery to the Emperor of China, who had the reputation of being a friend and patron of science. Leibniz hoped that in this manner the Chinese Emperor might be won over to Christianity.¹

In a letter² to Christian Wolff, Leibniz discusses the infinite series

$$1 + x + x^2 + x^3 + \dots,$$

which is obtained by dividing unity by $1 - x$. When x is less than 1, there is no trouble, the value of the series is equal to that of the quotient $1/1 - x$. For $x = -1$ the value of the quotient is $\frac{1}{2}$, while the series assumes the form

$$1 - 1 + 1 - 1 + 1 - 1 + \dots \text{ad infinitum.}$$

In this case, the infinite process in forming the series is devoid of any meaning; the whole argument breaks down. This is not what Leibniz and other contemporaries thought. He reasoned as follows: The series is infinite; when we take an even number of terms their sum is zero; for an odd number the sum is unity. Now, as we can not distinguish between an even and an odd infinite number of terms, the total number can be neither even nor odd, and the sum of all terms can therefore be obtained in the most natural manner by taking the average of the two values 0 and 1. This, according to Leibniz, gives the sum of $\frac{1}{2}$ for the infinite series. Leibniz refers to another Italian Jesuite, Grandi, who saw in this example the proof of creation as revealed in the Scriptures. Adding the terms of the series in groups of two, $(1 - 1) + (1 - 1) + (1 - 1) + \dots$, we get

$$0 + 0 + 0 + \dots \text{ad infinitum} = \frac{1}{2},$$

¹ La Place, "Théorie analytique des probabilités," Oeuvres, Vol. VII., pp. cxviii–cxix (1812).

² "Leibnizens mathematische Schriften," Erste Abteilung, Band IV., pp. 382–387 (Halle, 1850).

which, according to Grandi, proves the divine power to create something out of nothing. Laplace, commenting upon the fallacy of Leibniz' arguments, shows that there is nothing particular in the foregoing series, since the expansion of the quotient $(1+x)/(1+x+x^2)$ gives the same series, while its value for $x=1$ is two thirds.

The controversies on the infinite have lasted to the present day. It must be admitted, however, that they have assumed in most cases a very refined and exquisitely subtle form. In certain respects the difficulties have increased, and arguments in the style of Leibniz and his contemporaries appear as infantile efforts of a vanished period. The logicians have come to discuss the possibility of such concepts as "the class of all classes" and other artificial difficulties. Sometimes it would seem that the modern logicians are again occupying the stage vacated by the Greek school of Sophists centuries ago. It can not be said that they have come nearer the solution of the riddles of the universe.

Probably the greatest advance in the concept of infinity has been made by the definition that an aggregate of elements is infinite when the number of elements of a part of the aggregate is the same as the

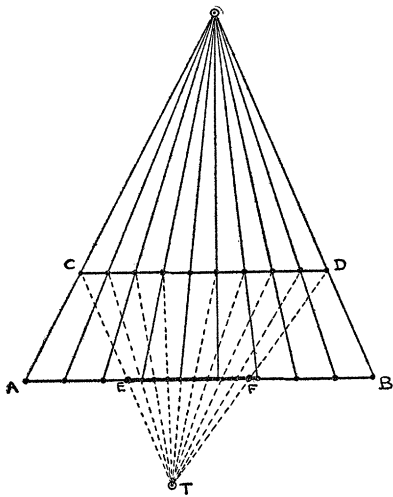


FIG. 1

number of all elements of the aggregate. A popular expression for this definition may be put in the form: Infinity has the property that a part is equal to the whole. Consider, for example, the series of all positive integers and multiply each by two, then we obtain a second series of integers that contains exactly as many integers as the first series:

1	2	3	4	5	
2	4	6	8	10	

The second series, however, is evidently contained in the first. The same definition applies to continuous sets of points which after the method of Dedekind may be put

into one-to-one correspondence with a continuous set of real numbers. Let AB , Fig. 1, be a segment of a straight line and consider the infinite number of all points of this segment, including A and B . From a point S project the points of AB upon the segment CD , so that in this correspondence there are as many points on CD as there are on AB . From another point T , we can project the points of CD upon the seg-

ment EF which lies on the same straight line as AB , such that AB contains EF as a part. Now there are as many points on EF as on CD in the projective correspondence, and consequently as many as on AB . In other words, the number of points on AB is the same as the number of a part of AB .

Pure logic, detached entirely from intuition, is impotent in the treatment of such problems. It is not only sterile, but engenders new antinomies.³

As has been stated before, there is no difficulty in the investigation of large numbers as long as they remain finite. Although the answers to problems which involve very large finite numbers may be amazing, they are definite and, at least theoretically, may be made intelligible through proper representation. It is evident that in concrete form a large number does not necessarily have to be connected with large spatial dimensions. Thus, in the relatively small space of one cubic centimeter of air under standard conditions, there are thirty million million millions of molecules. On the other hand, the distance of the nearest star (α centauri) is four and three tenths light years, or about twenty-five million million miles. There are telescopic stars whose distances are measured by thousands of light years. Light travels 186,330 miles a second and it would take the light of those stars thousands of years to reach the earth. To the scientific mind the revelations of astronomy are not more wonderful than those of the ultramicroscopic world. As an example we mention the remarkable molecular structure of crystals, as shown by Professor von Laue, who recently received the Nobel prize in physics for his discoveries. The wonderful geometric lattice-work formed by millions of molecules per cubic millimeter is revealed on the photographic plate by the diffraction of Röntgen rays on the molecules. Here we have the counterpart of the extremely large. The wave-length of Röntgen rays is so extremely small that they are diffracted by the extremely small molecules of the crystal. These may be considered as extremely small models of solar systems in which the electrons take the place of planets and other heavenly bodies.

Just as it is convenient to make use of auxiliary units like the astronomical unit (distance of sun from earth = 93 million miles) and the stellar unit (light year), great numbers in entirely different domains may be made more intelligible by concrete representation. Take, for example, the arithmetical expression 9^9 which means nine raised to a power whose exponent is nine raised to the ninth power. The number thus obtained is certainly finite, although its explicit representation would be well-nigh an impossible task. For practical purposes, and in

³ Poincaré, "Science et Méthode," pp. 210-214 (1908). See also "Les étapes de la philosophie mathématique," by Léon Brunschweig, pp. 370-393 (1912), where an excellent discussion of the logistic philosophy of mathematics is given.

the language of the layman, we might call this number infinite. It has 269 millions 690,000 places. To write it down, reserving one fourth of an inch for each cipher, it would take a strip of paper that would reach from Phoenix, Arizona, to New York City. Compared with this, the number of atoms in the Atlantic ocean would be negligibly small, and the Hindu fabulist with his story of a battle of a sextillion (unity with 36 ciphers annexed) monkeys would not strain our imagination as to largeness.

Large numbers, relatively small in comparison with those given above, are of great practical value in the theory of probabilities and statistics. By mathematical probability of an event we understand the ratio between the number of cases of actual occurrences and the total number of possible cases of the event. The greater the number of actual occurrences out of the total number, the larger the probability of such an event. When the two numbers are equal then the probability is equal to unity, in other words, the probable event becomes certain. A larger number of repetitions of a certain type of events contributes to our judgment of certainty. This fact has been stated by Poisson as "the law of large numbers," and may be expressed more definitely as follows: If p (a proper fraction) is the constant probability of an event and if the conditions for this event are repeated s times, whereby it occurs actually m times, then, by repeating the conditions a sufficiently large number of times, the difference between p and s/m can be made as small as we please. Thus, the probability that in the game of throwing a coin, head will appear is one half. Buffon, who once had the patience to do this 4,040 times, obtained 2,048 times head, so that the difference $p - s/m$ is in this case — 0.007. If Buffon had repeated the experiment a million times the difference would be very much smaller. The fact that during the last 6,000 years, days and nights have followed each other without a break over two million times makes it certain that the sun will rise in a clear sky or behind a cloud to-morrow. From the standpoint of mathematical probability and large numbers there is nothing extraordinary in two million confirmations of an event. And yet, in every-day life, such numbers in the realm of chance mean certainty to us. Mathematically speaking, we are satisfied with very crude approximations in this world. We never use anything else in practical life than comparatively very simple rational numbers. For $\pi = 3.1415926 \dots$, with an infinite number of ciphers, we never can use anything else than a rational approximation. Although the ten thousandth place of this transcendental number is an absolutely definite cipher between 0 and 9 (inclusive) we do not know what it is and it is a physically impossible task to determine it. We live in a domain of relatively small rational numbers and are satisfied with crude approximations where absolute accuracy can not be obtained.

Pure mathematical reasoning has nothing to do with the world of approximations. The number π has an absolutely definite meaning and exists. Likewise the incredibly large number defined as nine raised to the power (nine to the ninth power) is absolutely definite and finite in spite of the explosive character of its increase. The infinite series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \text{ad infinitum}$$

for which the sum of an increasing finite number of terms creeps along with unbelievable slowness, is divergent, *i. e.*, it has an infinite value. Indeed, when we add its terms successively in groups of one, two, four, eight, etc., we obtain an infinite number of such groups whose values are all larger than $\frac{1}{2}$, so that the value of the series itself is infinite. The problem shows an entirely different face when we attempt to actually add up the fractions in order to see how they contribute to the infinite value of the series. From the standpoint of the computer the legitimate mathematical claim for divergency of the series appears as a ridiculous pretense. Let us, for curiosity, figure out how many consecutive fractions of the series one must add to each other in order that their sum will amount to 100. Suppose that it is possible to write these fractions upon a strip of paper one half inch wide so that a cipher is reckoned for every half inch of length. With this strip the whole universe for which we assume a diameter of about 20,000 light years could be completely wrapped up and covered to a thickness of hundreds of miles. Millions of tons of ordinary writing paper would be necessary to manufacture such a strip of paper. This much is necessary to make a contribution of about one hundred units to the infinite account claimed by impeccable mathematical logic. In spite of practical impossibilities of this kind, there is of course no sound reason to dispute or criticize the rigorous conclusions of science as has been done in a very clever essay by Alexander Moszkowsky in "Zukunft."⁴

In conclusion, I shall briefly discuss a geometrical example involving an infinite process with a perfectly definite result. A complete geometrical representation of the process is impossible. But this is not more remarkable than the impossibility of making a visible representation of an actual mathematical point, or of a circle. In ancient times the possibility of a tangent was always connected with the idea of a continuous curve. Since the classic discovery of Weierstrass of continuous functions without derivatives, that idea had to be completely abandoned. As Peano and others have done, it is easy to state perfectly definite laws for the construction of continuous curves that admit of no definite tangents. To construct such a curve consider, for example, an equilateral triangle whose sides are equal to the unit of length, Fig. 2. Divide each side into three equal parts, remove the middle parts and

⁴ "Das Geheimniss der grossen Zahl," Vol. 78, pp. 85-94 (1912).

bridge the gap over by two adjacent pieces, each of one third the length of the original side. In this manner a new Fig. 3 is obtained whose circumference is evidently four thirds of the original length of three units. We now repeat the same operation on each of the twelve segments of the new figure and thus obtain Fig. 4, whose circumference is four thirds of that of Fig. 3, *i. e.*, $(\frac{4}{3})^2 \cdot 3$. We repeat this process a

FIG. 2

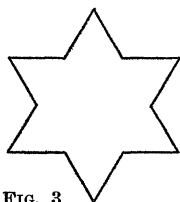
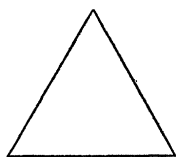


FIG. 3

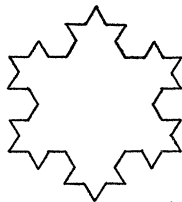


FIG. 4

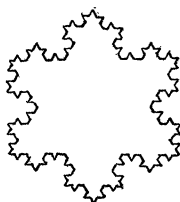


FIG. 5

third (Fig. 5), fourth time, and so forth, *ad infinitum*. The circumference of each new figure is always four thirds of that of the previous figure. Repeating the process n times, we obtain a crinkly curve with the length $(\frac{4}{3})^n \cdot 3$. When we let n approach infinity, the figure approaches a perfectly definite continuous curve of finite dimensions, but of infinite length, since the n th power of $\frac{4}{3}$ increases indefinitely as n increases indefinitely. This curve defined by a very simple geometric law is a so-called Jordan-curve and has H. v. Koch and Cesaro as its originators.⁵ A complete geometric construction of this curve, involving the configuration of the limit, is impossible.

As is shown by this and other examples, we can conceive in a perfectly definite mathematical sense infinite processes which do not admit of complete explicit representation. This, however, is not more remarkable than the impossibility of representing explicitly all elements of certain extremely large finite and transcendental numbers.

Insurmountable difficulties only appear when we try to grasp problems involving the infinite by introducing peculiar logistic postulates. Then paradoxes and antinomies become supreme.

⁵ E. Study, "Konforme Abbildungen," pp. 42-43 (1913).